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A Class of Cardinal Splines with Hermite Type Interpolation

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1. INTRODUCTION

Recently Lipow and Schoenberg [4] have studied the case of cardinal splines of degree n with values and the first $r-1$ derivatives prescribed at the integers. This leads them to a study of the zeros of a polynomial $\Pi_{n,r}(\lambda)$ which is related to the characteristic polynomial $P - \lambda I = \|\binom{i}{j} - \lambda\delta_{ij}\|$ of the Pascal matrix $P = \|\binom{i}{j}\|$ $(i, j = 0, 1,...)$. They use a well-known theorem of Gantmacher and Krein on the simplicity of the eigenvalues of oscillating matrices to prove that $\Pi_{n,r}(\lambda)$ has real simple zeros of sign $(-1)^r$. This result is then applied to solve the interpolation problem.

Here we propose to study the problem when the cardinal spline of degree n has its values and first $r-1$ derivatives prescribed at the points $\theta + \nu$, $(\nu = 0, \pm 1, \pm 2,...)$ where $0 \leqslant \theta < 1$ is a fixed number. For $\theta = 0$ we get the results of Lipow and Schoenberg [4]. As in the case of Lipow and Schoenberg, the main difficulty is to prove that the zeros of the polynomial $\Pi_{n,r}(\theta; \lambda)$ are real and simple. Our method is based on a determinantal identity (see [1, p. 7]) instead of the Gantmacher and Krein Theorem and is similar to that in [3].

In Section 2 we state the problem of interpolation and the main theorem. The eigensplines and eigenvalues are dealt with in Section 3 where the polynomial $\Pi_{n,r}(\theta; \lambda)$ is explicitly defined. In Section 4 we give a proof of the main Theorem 1 which is based on Theorems 2 and 3 ,which are proved in Sections 7 and 8. Some useful identities are proved in Section 5, while Section 6 deals with the Hankel determinant of the exponential Euler polynomials $A_n(\theta; \lambda)$ and their relation to the polynomials $\Pi_{n,r}(\theta; \lambda)$.

2. Statement of the Problem

Let n, r be natural numbers with $n \geqslant r$ and suppose we are prescribed r bi-infinite sequences of data

$$y^{(s)} = (y_\nu^{(s)}) \qquad (s = 0, 1,..., r - 1). \tag{2.1}$$

Let $\mathscr{S}_{n,r}$ denote the class of cardinal spline functions $S(x)$ of degree n such that $S(x) \in C^{n-r}(-\infty, \infty)$.

Problem I. To find $S(x) \in \mathscr{S}_{n,r}$ such that

$$S^{(s)}(\nu + \theta) = y_\nu^{(s)} \qquad (s = 0, 1,..., r - 1)\ \forall \text{ integers } \nu, \tag{2.2}$$

where $0 \leqslant \theta < 1$.

We can easily prove the following lemma.

Lemma 1. *If $0 < \theta < 1$, the interpolation Problem* I *always has solutions which form a linear manifold in $\mathscr{S}_{n,r}$ of dimension $n - r + 1$. For $\theta = 0$, the linear manifold is of dimension $n - 2r + 1$.*

The proof follows the same pattern as in Lipow and Schoenberg [4].

Our object is to prove

Theorem 1. *If θ is not a zero of the polynomial $\Pi_{n,r}(x; (-1)^r)$ and if the data* (2.1) *satisfy*

$$y_\nu^{(s)} = 0(|\nu|^\gamma) \qquad (s = 0, 1, 2,..., r - 1) \tag{2.3}$$

for some $\gamma > 0$, then there exists a unique spline $S(x) \in \mathscr{S}_{n,r}$ satisfying (2.2) *such that*

$$S(x) = O(|x|^\gamma). \tag{2.4}$$

3. Eigensplines and Characteristic Polynomials

We shall denote the null space of $\mathscr{S}_{n,r}$ with respect to Problem I by $\mathring{\mathscr{S}}^\theta_{n,r}$, which is defined by

$$\mathring{\mathscr{S}}^\theta_{n,r} = \{S(x) \in \mathscr{S}_{n,r}: S^{(s)}(\nu + \theta) = 0\ (s = 0, 1,..., r - 1)\ \forall \text{ integers } \nu\}. \tag{3.1}$$

The eigensplines of $\mathscr{S}_{n,r}$ corresponding to this problem are those splines $S(x) \in \mathring{\mathscr{S}}^\theta_{n,r}$ such that

$$S(x + 1) = \lambda S(x) \qquad (x \in \mathbb{R}) \quad \lambda \neq 0. \tag{3.2}$$

We shall call λ the corresponding eigenvalue.

LEMMA 2. *If $S(x)$ is an eigenspline in $\mathring{\mathscr{S}}^{\theta}_{n,r}$ with eigenvalue λ, then $\mathscr{S}(x) = S(-x)$ is an eigenspline in $\mathring{\mathscr{S}}^{1-\theta}_{n,r}$ with eigenvalue λ^{-1}.*

We omit the proof, which is almost obvious.

Suppose $S(x) \in \mathring{\mathscr{S}}^{\theta}_{n,r}$ is an eigenspline satisfying the functional relation (3.2), and suppose $P(x)$ is the polynomial component of $S(x)$ in $[0, 1]$. Then

$$\begin{aligned} &P^{(s)}(1) = \lambda P^{(s)}(0) \qquad (s = 0, 1,..., n - r), \\ &P^{(s)}(\theta) = 0 \qquad\qquad\;\; (s = 0, 1,..., r - 1). \end{aligned} \tag{3.3}$$

If we set $P(x) = a_0 x^n + \binom{n}{1} a_1 x^{n-1} + \cdots + a_n$, the coefficients $a_0, a_1,..., a_n$ are to be determined by (3.3), which gives a homogeneous system of equations whose determinant is $\Pi_{n,r}(\theta; \lambda)$, where

$$\Pi_{n,r}(\theta; \lambda) =$$

$$\begin{vmatrix}
1 & \binom{r}{1} & \cdots \binom{r}{r-1}(1-\lambda) & 0 & \cdots\cdots & 0 \\
1 & \binom{r+1}{1} & \cdots\cdots & \binom{r+1}{r}(1-\lambda) & 0 \cdots\cdots & 0 \\
\vdots & \vdots & & & & \vdots \\
1 & \binom{n-1}{1} & \cdots\cdots & \cdots\cdots & (1-\lambda) & 0 \\
1 & \binom{n}{1} & \cdots\cdots & \cdots\cdots & \cdots\cdots & (1-\lambda) \\
\theta^{n-r+1} & \binom{n-r+1}{1}\theta^{n-r} & \cdots\cdots & 1 & 0 \cdots\cdots & 0 \\
\vdots & \vdots & & & & \vdots \\
\theta^{n-1} & \binom{n-1}{1}\theta^{n-2} & \cdots\cdots & \cdots\cdots & 1 & 0 \\
\theta^{n} & \binom{n}{1}\theta^{n-1} & \cdots\cdots & \cdots\cdots & \binom{n}{n-1}\theta & 1
\end{vmatrix}. \tag{3.4}$$

Observe that $\Pi_{n,r}(\theta; \lambda)$ is a polynomial of degree $n - r + 1$ in λ and of degree $r(n - r + 1)$ in θ.

From (3.4) it is easy to see that if $r \leqslant n \leqslant 2r - 1$, we have

$$\begin{aligned} &\Pi_{n,r}(0; \lambda) = \Pi_{n,r}(\lambda) = 1, \\ &\Pi_{n,r}(1; \lambda) = (-1)^{(n+1)(n-r+1)} \lambda^{n-r+1}. \end{aligned} \tag{3.5}$$

When $n \geqslant 2r - 1$

$$\begin{aligned} &\Pi_{n,r}(0; \lambda) = \Pi_{n,r}(\lambda), \\ &\Pi_{n,r}(1, \lambda) = \lambda^{r} \Pi_{n,r}(\lambda). \end{aligned} \tag{3.6}$$

THEOREM 2. *If* $0 < \theta < 1$ *and* $n \geqslant r$, *the polynomial* $\Pi_{n,r}(\theta; \lambda)$ *as a polynomial in* λ *has only real zeros, all simple and of sign* $(-1)^r$. *Furthermore the zeros of* $\Pi_{n,r}(\theta; \lambda)$ *and* $\Pi_{n-1,r}(\theta; \lambda)$ *interlace.*

Remark 1. If $\theta = 0$ or $\frac{1}{2}$, it follows from Lemma 2 that $\Pi_{n,r}(\theta; \lambda)$ is a *reciprocal polynomial in* λ.

THEOREM 3. *The polynomial* $\Pi_{n,r}(\theta; (-1)^r)$ *is of degree* $r(n - r + 1)$ *in* θ *and has exactly* $n - r + 1$ *zeros, all simple, in* $(0, 1)$ *if* $r \leqslant n \leqslant 2r - 1$. *For* $n \geqslant 2r - 1$, *this polynomial has exactly* $r - 1$ *(or* r*) simple zeros in* $(0, 1)$ *according as* n *is even or odd.*

Remark 2. If $r = 1$ and n is odd the only zero of $\Pi_{n,r}(\theta; (-1)^r)$ in $(0, 1)$ is $\frac{1}{2}$. If n is even and $r = 1$, this polynomial has no zero in $(0, 1)$ but $\theta = 0$ is a zero. In fact it can be shown that $\Pi_{n,1}(\theta; -1) = (-2)^n E_n(\theta)$ where $E_n(\theta)$ is the classical Euler polynomial.

Remark 3. If $\theta = \frac{1}{2}$, it follows from Remark 1 that $\Pi_{n,r}(\frac{1}{2}; \lambda)$ is a reciprocal polynomial of degree $n - r + 1$. Thus $\Pi_{n,r}(\frac{1}{2}; (-1)^r) = 0$ or $\neq 0$ according as $n - r + 1$ is odd or even. It follows from Theorem 1 that if $\theta = \frac{1}{2}$ the interpolatory spline is unique if n and r have different parity.

Remark 4. It follows from (3.6) that if n is even, $\Pi_{n,r}(0; (-1)^r) = \Pi_{n,r}(1; (-1)^r) = 0$.

4. SOLUTION OF THE INTERPOLATION PROBLEM

We shall require Theorems 2 and 3 for the solution of the interpolation problem formulated in Section 2. By Theorem 2 we know that the zeros λ_j $(j = 1, 2, ..., n - r + 1)$ of $\Pi_{n,r}(\theta; \lambda)$ are ditsinct. The eigensplines $S_j(x)$ are constructed from λ_j such that

$$S_j(x + 1) = \lambda_j S_j(x) \qquad (x \in \mathbb{R}). \tag{4.1}$$

Since the λ_j's are distinct $S_j(x)$ $(j = 1, 2, ..., n - r + 1)$ are linearly independent and so we have the following:

LEMMA 3. *If* $S(x) \in \mathcal{S}^{\theta}_{n,r}$, *then there is a unique representation*

$$S(x) = \sum_{j=1}^{n-r+1} c_j S_j(x) \tag{4.2}$$

with appropriate constants c_j.

We are now ready to prove Theorem 1. By assumption on θ, $\Pi_{n,r}(\theta; (-)^r) \neq 0$, hence no eigenvalues lie on the unit circle. Suppose

$$\begin{aligned} |\lambda_j| &< 1 \quad \text{for} \quad j = 1, 2, \dots, k, \\ |\lambda_j| &> 1 \quad \text{for} \quad j = k+1, \dots, n-r+1. \end{aligned} \tag{4.3}$$

Following an argument similar to that of Lipow and Schoenberg [4, p. 291] we may set

$$L_s(x) = \begin{cases} \sum_{j=1}^{k} c_j S_j(x) & (x \geqslant 1), \\ \sum_{j=k+1}^{n-r+1} d_j S_j(x) & (x \leqslant 0), \\ P(x) & (0 \leqslant x \leqslant 1), \end{cases} \tag{4.4}$$

where

$$P(x) = a_0(x-\theta)^n + a_1(x-\theta)^{n-1} + \cdots + a_{n-r}(x-\theta)^r + (x-\theta)^s/s!\,. \tag{4.5}$$

If $L_s(x) \in \mathscr{S}_{n,r}$, the continuity requirements at 0 and 1 give the relations

$$\left.\begin{aligned} P^{(s)}(1) &= \sum_{j=1}^{k} c_j S_j^{(s)}(1), \\ P^{(s)}(0) &= \sum_{j=k+1}^{n-r+1} d_j S_j^{(s)}(0) \end{aligned}\right\} \quad (s = 0, 1, \dots, n-r). \tag{4.6}$$

This is a nonhomogeneous linear system of $2(n-r+1)$ equations in $2(n-r+1)$ unknowns $\{c_j\}_1^k$, $\{d_j\}_{k+1}^{n-r+1}$ and $\{a_j\}_0^{n-r}$.

In order to prove that this linear system is nonsingular, we consider the homogeneous system corresponding to system (4.6). The existence of a nontrivial solution of this homogeneous system implies the existence of a nonzero spline $S(x) \in \mathscr{S}_{n,r}^{\theta} \cap L_{1,r}$. Since every $S(x) \in \mathscr{S}_{n,r}^{\theta}$ can be written as a linear combination of the eigensplines $S_j(x)$ which have different orders of exponential growth at least on one side of the real axis, it follows from the fact that $S(x) \in L_{1,r}$, that $S(x) \equiv 0$. This proves that the linear system (4.6) is nonsingular.

The solution of the interpolation problem can then be given explicitly by

$$\begin{aligned} S(x) = {} & \sum_{-\infty}^{\infty} y_\nu L_0(x-\theta-\nu) + \sum_{-\infty}^{\infty} y_\nu' L_1(x-\theta-\nu) \\ & + \cdots + \sum_{-\infty}^{\infty} y_\nu^{(r-1)} L_{r-1}(x-\theta-\nu). \end{aligned}$$

Since the fundamental splines and their derivatives are of exponential decay as $|x| \to \pm\infty$, it follows by imitating the reasoning in Lipow and Schoenberg [4] that the above representation of the interpolatory spline is unique when the data is of power growth. This completes the proof of Theorem 1.

5. Some Identities

In order to prove Theorems 2 and 3 we shall need some identities which we formulate as lemmas.

Lemma 4. *Let n, r be positive integers with $n \geqslant r + 1$. Then the following identities hold*:

$$r\Pi_{n,r+1}(\theta;\lambda)\,\Pi_{n-2,r-1}(\theta;\lambda)$$
$$= n\{\Pi_{n-1,r}(\theta;\lambda)\}^2 - (n-r)\,\Pi_{n-2,r}(\theta;\lambda)\,\Pi_{n,r}(\theta;\lambda), \tag{5.1}$$
$$r\Pi_{n,r+1}(\theta;\lambda)\,\Pi_{n-1,r-1}(\theta;\lambda)$$
$$= -\Pi'_{n,r}(\theta;\lambda)\,\Pi_{n-1,r}(\theta;\lambda) + \Pi_{n,r}(\theta;\lambda)\,\Pi'_{n-1,r}(\theta;\lambda), \tag{5.2}$$

where the prime denotes derivative with respect to θ.

Proof. The proof depends on the determinantal identity [1, p. 7]

$$\begin{vmatrix} D(\mathbf{a},\mathbf{c},\mathbf{f}) & D(\mathbf{a},\mathbf{d},\mathbf{f}) \\ D(\mathbf{b},\mathbf{c},\mathbf{f}) & D(\mathbf{b},\mathbf{d},\mathbf{f}) \end{vmatrix} = D(\mathbf{a},\mathbf{b},\mathbf{f})\,D(\mathbf{c},\mathbf{d},\mathbf{f}),$$

where $D(\mathbf{a},\mathbf{b},\mathbf{f}) \equiv D(\mathbf{a},\mathbf{b},\mathbf{f}^{(1)},\dots,\mathbf{f}^{(n)})$ is the determinant formed from the column vectors $\mathbf{a},\mathbf{b},\mathbf{f}^{(1)},\dots,\mathbf{f}^{(n)} \in \mathbb{R}^{n+2}$. The techniques are the same as in [3] (see also [2]) and we shall omit them here. ∎

The special case of (5.1) when $\theta = 0$ is given in [3].

Lemma 5. *If $n \geqslant r$, then*

$$\Pi_{n,r}(\theta;0) = (1-\theta)^{r(n-r+1)}. \tag{5.3}$$

Proof. It is clear that if $r = 0$, (5.3) is valid. From the generating function (see [7])

$$(\lambda - 1)\,e^{\theta z}/(\lambda - e^z) = \sum_{n=0}^{\infty} [A_n(\theta;\lambda)/n!]\,z^n \tag{5.4}$$

it is easy to see that

$$A_n(\theta;0) = (\theta-1)^n, \tag{5.5}$$

and from (3.4) we have

$$\Pi_{n,1}(\theta; \lambda) = (\lambda - 1)^n A_n(\theta; \lambda). \tag{5.6}$$

Hence $\Pi_{n,1}(\theta; 0) = (1 - \theta)^n$, which proves (5.3) for $r = 1$.

The identity (5.3) now follows on using the identity (5.1) for $\lambda = 0$, by easy induction on r.

6. Hankel Determinants of Exponential Euler Polynomials

The polynomial $\Pi_{n,r}(\theta; \lambda)$ is closely related to the exponential Euler polynomials $A_n(\theta; \lambda)$ generated by the relation (5.4) (see [7]). Set

$$H_r(a_n) = \begin{vmatrix} a_n & a_{n-1} & \cdots & a_{n-r+1} \\ a_{n-1} & a_{n-2} & \cdots & a_{n-r} \\ \vdots & & & \vdots \\ a_{n-r+1} & a_{n-r} & \cdots & a_{n-2r+2} \end{vmatrix}. \tag{6.1}$$

Then the following relation holds (see also [2]).

Theorem 4. *Let $r = 1, 2, \ldots$. For $n \geqslant 2r - 1$, we have*

$$H_r\left(\frac{A_n(\theta; \lambda)}{n!}\right) = c(n, r)\frac{\Pi_{n,r}(\theta; \lambda)}{(1 - \lambda)^{n-r+1}} \tag{6.2}$$

where

$$c(n, r) = \frac{(-1)^{nr+[r/2]}\, 1!2!\ldots(r - 1)!}{n!(n - 1)!\ldots(n - r + 1)!}.$$

The proof follows from the identity (5.1) by using induction on r and is exactly the same as for the special case when $\theta = 0$ given in [3] (see also [2]).

7. Proof of Theorem 2

We shall first prove the theorem for $n = r$ and $n = r + 1$. We first observe that $\Pi_{n,r}(\theta; \lambda)$ is a polynomial of degree $n - r + 1$ in λ and that the coefficient of λ^{n-r+1} is

$$(-1)^{(r+1)(n-r+1)} \begin{vmatrix} \theta^{n-r+1} & \binom{n-r+1}{1}\theta^{n-r} & \ldots & \binom{n-r+1}{r-1}\theta^{n-2r+2} \\ \theta^{n-r+2} & \binom{n-r+2}{1}\theta^{n-r+1} & \ldots & \binom{n-r+2}{r-1}\theta^{n-2r+3} \\ \vdots & \vdots & & \vdots \\ \theta^{n} & \binom{n}{1}\theta^{n-1} & \ldots & \binom{n}{r-1}\theta^{n-r+1} \end{vmatrix}.$$

Using the fact that $P\binom{n-r+1,\ldots,n}{0,1,\ldots,r-1} = 1$ (see [4]) it is easy to see that

$$\Pi_{n,r}(\theta; \lambda) = \{(-1)^{r+1}\lambda\}^{n-r+1}\, \theta^{r(n-r+1)} + \text{lower order terms}. \tag{7.1}$$

From Lemma 5 and (7.1) we see that $\Pi_{r,r}(\theta; \lambda)$ is a first-degree polynomial in λ given by

$$\Pi_{r,r}(\theta; \lambda) = (1-\theta)^r + (-1)^{r+1}\, \theta^r \lambda. \tag{7.2}$$

Hence for $0 < \theta < 1$, $\Pi_{r,r}(\theta; \lambda)$ has exactly one zero $\lambda_1^{(r)}$ of sign $(-1)^r$. It follows from (5.1) of Lemma 1 that $r\Pi_{r+1,r+1}(\theta; \lambda_1^{(r)})\, \Pi_{r-1,r-1}(\theta; \lambda_1^{(r)}) = -\Pi_{r+1,r}(\theta; \lambda_1^{(r)})$ so that $\Pi_{r+1,r}(\theta; \lambda_1^{(r)}) < 0$. Since $\Pi_{r+1,r}(\theta; 0) > 0$ by Lemma 2 and $\Pi_{r+1,r}(\theta; \lambda) > 0$ as $|\lambda| \to \infty$ by (7.1), it follows that $\Pi_{r+1,r}(\theta; \lambda)$ has exactly two zeros, one between 0 and λ_1 and the other between λ_1 and ∞ (or $-\infty$) when r is even (or odd).

This proves the theorem for $n = r$ and $n = r + 1$. The rest of the proof follows by induction on n using the identity (5.1). ∎

8. Proof of Theorem 3

Again our proof will proceed by induction on n. We shall assume that r is odd, since the case for even r can be treated similarly.

From (7.2) it follows that $\Pi_{r,r}(\theta; -1) = (1-\theta)^r - \theta^r$, $\Pi_{r+1,r+1}(\theta; -1) = (1-\theta)^{r+1} + \theta^{r+1}$ and $\Pi_{r-1,r-1}(\theta; -1) = (1-\theta)^{r-1} + \theta^{r-1}$, so that (5.1) gives

$$\begin{aligned}\Pi_{r+1,r}(\theta; -1) &= (r+1)[(1-\theta)^r - \theta^r]^2 \\ &\quad - r[(1-\theta)^{r+1} + \theta^{r+1}][(1-\theta)^{r-1} + \theta^{r-1}].\end{aligned} \tag{8.1}$$

The polynomial $\Pi_{r,r}(\theta; -1)$ obviously vanishes for $\theta = \frac{1}{2}$ and is a decreasing function of θ in $(0, 1)$. The polynomial $\Pi_{r+1,r}(\theta; -1)$ is symmetric about $\theta = \frac{1}{2}$ and positive for $\theta = 1$ and negative for $\theta = \frac{1}{2}$. An easy computation shows that $\Pi'_{r+1,r}(\theta; -1)$ is positive for $\frac{1}{2} < \theta < 1$. This shows that $\Pi_{r+1,r}(\theta; -1)$ has exactly two simple zeros in $(0, 1)$. This proves the theorem for $n = r$ and $r + 1$.

The proof can now be completed by induction on n using the identities (5.1) and (5.2) and the relations (3.5) and (3.6). Indeed if we denote the zeros of $\Pi_{n-1,r}(\theta; -1)$ in the interval $(0, 1)$ by $\theta_j^{(n-1)}$ $(j = 1, 2,\ldots, n-r)$ so that

$$0 < \theta_1^{(n-1)} < \theta_2^{(n-1)} < \cdots < \theta_{n-r}^{(n-1)} < 1, \tag{8.2}$$

then (5.1) and (3.5) show that for $r \leqslant n \leqslant 2r - 1$, $\Pi_{n,r}(\theta; -1)$ has at least one zero in each of the intervals $(0, \theta_1^{(n-1)})$, $(\theta_{n-r}^{(n-1)}, 1)$ and $(\theta_j^{(n-1)}, \theta_{j+1}^{(n-1)})$

($j = 1, 2, \ldots, n - r - 1$). Using (5.2) we conclude that $\Pi_{n,r}(\theta; -1)$ has exactly one zero in each of these intervals.

Similarly for $n \geqslant 2r - 1$, using (5.1), (3.6), and (5.2) we conclude that $\Pi_{n,r}(\theta; -1)$ has exactly $(r - 1)$ zeros in (0, 1) together with 0 and 1 if n is even, and has exactly r zeros in (0, 1) if n is odd. ∎

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